COMPUTING SPECIAL JORDAN PENCILS

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Abstract

In the study of composite matrials, knowing only the composition of a material does not give us exact figures of the properties, such as elasticity, malleablity, and conductivity, of the material, but only inequalities. However, in some cases, the composition alone is sufficient to determine some relations between the components of the tensor of effective properties of a composite. Remarkably, these special cases are in 1-1 correspondence with peculiar algebraic objects that resemble Jordan algebras. This paper focuses on solving an interesting theoretical mathematical problem that arises in the context of the Hall effect for fiber-reinforced composites, namely how to compute these algebraic objects that we call special Jordan pencils for low-dimensional cases.

1 Introduction

Let $\mathrm{End}(\mathbb{R}^n)$ denote the space of all real $n \times n$ matrices, and let $\mathcal{A}$ be a subspace of $\mathrm{End}(\mathbb{R}^n)$.

Definition 1.1 *We call a subspace* $\Pi \subset End(\mathbb{R}^n)$ *a special Jordan pencil of* $(End(\mathbb{R}^n), \mathcal{A})$ *if*

$$\mathbf{K_1 A K_2} + \mathbf{K_2 A K_1} \in \Pi \tag{1}$$

for all $\mathbf{K_1}, \mathbf{K_2} \in \Pi$*, and all* $\mathbf{A} \in \mathcal{A}$*.*

We shall also use this notation to denote the set of all special Jordan pencils. In other words, if Π is a special Jordan pencil of $(\mathrm{End}(\mathbb{R}^n), \mathcal{A})$, then we write $\Pi \in (\mathrm{End}(\mathbb{R}^n), \mathcal{A})$. We call the property in (1) the *two-chain condition.* The next lemma is an equivalent property that will be very useful throughout this paper.

Lemma 1.2 *A subspace* Π *satisfies the two-chain condition if and only if*

$$\mathbf{KAK} \in \Pi \tag{2}$$

for all $\mathbf{K} \in \Pi$ *and all* $\mathbf{A} \in \mathcal{A}$*.*

Proof: This condition is clearly necessary. It is sufficient because letting $\mathbf{K} = \mathbf{K_1} + \mathbf{K_2}$, we see that

$$(\mathbf{K_1} + \mathbf{K_2})\mathbf{A}(\mathbf{K_1} + \mathbf{K_2}) = \mathbf{K_1AK_1} + \mathbf{K_2AK_2} + \mathbf{K_1AK_2} + \mathbf{K_2AK_1} \in \Pi.$$

By assumption, $\mathbf{K_1AK_1}$ and $\mathbf{K_2AK_2}$ are in Π, so $\mathbf{K_1AK_2} + \mathbf{K_2AK_1} \in \Pi$. □

Define $\mathcal{A}_2$ as the set of all 2×2 symmetric trace-free matrices. That is,

$$\mathcal{A}_2 = \left\{ \begin{bmatrix} s & t \\ t & -s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Clearly, the subspace $\{\mathbf{0}\}$ and $\mathrm{End}(\mathbb{R}^2)$ are special Jordan pencils of $(\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$. The next theorem gives a complete description of all non-trivial special Jordan pencils of $(\mathrm{End}(\mathbb{R}^2), \mathcal{A}^2\}$.

Theorem 1.3 *If a non-empty proper subspace $\Pi \subset End(\mathbb{R}^2)$ is a special Jordan pencil of $(End(\mathbb{R}^2), \mathcal{A}_2)$, then Π is a subspace from the following list:*

1. $\mathcal{A}_2$
2. $\Pi_\beta = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = \beta(b - c) \right\}$, $\beta \neq 0$
3. $\Pi_{\mathbf{a}} = \{\mathbf{v} \otimes \mathbf{a} \mid \mathbf{v} \in \mathbb{R}^2\}$
4. $\Pi_{\mathbf{a}}^T$
5. $\Pi_{\mathbf{a},\mathbf{b}} = \mathbb{R}(\mathbf{a} \otimes \mathbf{b})$ *(where $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T$).*

Notice that if $\beta = 0$, then Π_β is the space of trace-free matrices. Also, if β is ∞, or more precisely, if $(b - c)/(a + d) = 0$, then Π_β is the space of symmetric matrices. Also, for $\beta_1 \neq \beta_2$, the subspace $\Pi_{\beta_1} \cap \Pi_{\beta_2} = \mathcal{A}_2$. In fact, all subspaces from the list in theorem 1.3 can be generated by taking intersections and transposes of subspaces from items 2 and 3.

Theorem 1.3 will not be proven here; a proof can be found in [1]. The focus of this paper is on expanding this result to $\mathrm{End}(\mathbb{R}^3)$. In particular, we want to find all special Jordan pencils of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$, where

$$\mathcal{A} = \left\{ \begin{bmatrix} s & t & 0 \\ t & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}. \tag{3}$$

In this paper it will be necessary to compute some matrices of the form $\mathbf{KAK}$, and $\mathbf{K_1AK_2} + \mathbf{K_2AK_1}$, for arbitrary matrices $\mathbf{A} \in \mathcal{A}$, and where $\mathbf{K}, \mathbf{K_1}$, and $\mathbf{K_2}$ are matrices in a given subspace. To simplify notation, we shall define

$$\mathbf{K}^{\#} = \mathbf{KAK}$$

and

$$\mathbf{K_1}\#\mathbf{K_2} = \mathbf{K_1}\mathbf{A}\mathbf{K_2} + \mathbf{K_2}\mathbf{A}\mathbf{K_1}.$$

Though we use the # symbol to denote both a uniary and a binary operation, the indended usage should be clear from the context.

2 Subspace decomposition

Henceforth, we shall use Π_2 and Π_3 to denote an arbitrary special Jordan pencil of $(\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$, and $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$, respectively. For convenience, we will denote 3×3 matrices as 2×2 block-matrices:

$$\begin{bmatrix} \mathbf{K} & \mathbf{u} \\ \mathbf{v} & \lambda \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & u_1 \\ k_{21} & k_{22} & u_2 \\ v_1 & v_2 & \lambda \end{bmatrix}$$

Define $P_{11} : \Pi \to \Pi_2$ by

$$P_{11} : \begin{bmatrix} \mathbf{K} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} \longmapsto \mathbf{K}.$$

It is easy to check that if $\Pi \in (\mathrm{End}(\mathbb{R}^3), \mathcal{A})$, then $P_{11}(\Pi) \in (\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$. It remains to determine all possibilities for the subspace $(\mathbf{u}, \mathbf{v}, \rho) \subset \mathbb{R}^5$.

Since P_{11} is $\mathbb{R}$-linear, $\Pi \cong P_{11}(\Pi) \oplus \ker P_{11}$. It follows that the natural quotient map $\overline{P}_{11} : \Pi/\ker P_{11} \to \Pi_2$ is injective. Here, $\overline{P}_{11}$ sends a coset of Π to an element in Π_2. Any subspace $\Pi \subset \mathrm{End}(\mathbb{R}^3)$ can be decomposed orthogonally as $\Pi \cong \ker P_{11} \oplus (\ker P_{11})^{\perp}$, where

$$(\ker P_{11})^{\perp} \cong \Pi/\ker P_{11} \cong P_{11}(\Pi) = \Pi_2.$$

Therefore, the subspaces $\ker P_{11}$ and $(\ker P_{11})^{\perp}$ are

$$\begin{aligned} \ker P_{11} &= \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} : (\mathbf{u}, \mathbf{v}, \rho) \subset \mathbb{R}^5 \right\} \\ (\ker P_{11})^{\perp} &= \left\{ \begin{bmatrix} \mathbf{K} & w(\mathbf{K}) \\ q(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix} : \mathbf{K} \in \Pi_2 \right\} \end{aligned} \tag{4}$$

where $w, q : \Pi_2 \to \mathbb{R}^2$ and $\lambda : \Pi_2 \to \mathbb{R}$ are $\mathbb{R}$-linear functions. In order to find all 3×3 special Jordan pencils, we must find possible linear maps w, q, and λ, and all subspaces $\ker P_{11}$.

3 Determining the linear maps

The next goal is to determine all possible $\mathbb{R}$-linear maps w and q satisfying (4). A simple calculation shows that

$$\begin{bmatrix} \mathbf{K} & w(\mathbf{K}) \\ q(\mathbf{K}) & \rho \end{bmatrix}^{\#} = \begin{bmatrix} \mathbf{KAK} & \mathbf{KA}w(\mathbf{K}) \\ \mathbf{K}^T\mathbf{A}q(\mathbf{K}) & \mathbf{A}w(\mathbf{K}) \cdot \mathbf{A}q(\mathbf{K}) \end{bmatrix}.$$

Therefore, it is sufficient to find all linear maps w that satisfy

$$w(\mathbf{KAK}) = \mathbf{KA}w(\mathbf{K}) \tag{5}$$

and all linear maps q satisfying

$$q(\mathbf{KAK}) = \mathbf{K}^T\mathbf{A}q(\mathbf{K}), \tag{6}$$

for all $\mathbf{K} \in \Pi_2$, $\mathbf{A} \in \mathcal{A}_2$.

Lemma 3.1 *The $\mathbb{R}$-linear map $w : \mathbb{R}^2 \to \mathbb{R}$ satisfies (5) if and only if the map q defined by $q(\mathbf{K}) = w(\mathbf{K}^T)$ satisfies (6).*

Proof: If $w(\mathbf{K}) = q(\mathbf{K}^T)$, then

$$q(\mathbf{KAK}) = w(\mathbf{K}^T\mathbf{AK}^T) = \mathbf{K}^T\mathbf{A}w(\mathbf{K}^T) = \mathbf{K}^T\mathbf{A}q(\mathbf{K}).$$

The proof of the converse is almost identical and will be omitted. □

Lemma 3.1 implies that we don't need to determine both w and q; finding all possiblibities for one will suffice. We shall turn our attention to finding w, which is the result of the next theorem.

Theorem 3.2 *If $w : \mathbb{R}^2 \to \mathbb{R}$, and $w(\mathbf{KAK}) = \mathbf{KA}w(\mathbf{K})$ for all $\mathbf{K} \in \Pi_2$, $\mathbf{A} \in \mathcal{A}_2$, then $w(\mathbf{K}) = \mathbf{Kr}$ for some fixed $\mathbf{r} \in \mathbb{R}^2$.*

Proof: The result needs to be proven separately for each of the six non-empty families of 2×2 special Jordan pencils from Theorem 1.3. Not all cases will be shown here, for most just consist of lots of calculations and not too many deep concepts. However, we shall prove the cases where $\Pi = \mathcal{A}_2$ and where $\Pi = \mathrm{End}(R^2)$.

First, suppose that $\Pi = \mathcal{A}_2$. Let $\mathbf{I}$ be the identity matrix, and define

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{7}$$

Let $\mathbf{r} = w(\mathbf{I})$. For any $\mathbf{A} \in \mathcal{A}$, the product $\mathbf{SAS} = \mathbf{A}$. Therefore,

$$\mathbf{SA}w(\mathbf{S}) = w(\mathbf{SAS}) = w(\mathbf{A}) = \mathbf{IA}w(\mathbf{I}) = \mathbf{Ar}.$$

This proves the theorem for $\Pi = \mathcal{A}_2$. To prove it for $\Pi = \mathrm{End}(\mathbb{R}^2)$, we first recognize that

$$\mathrm{End}(\mathbb{R}^2) \cong \mathbb{R}\mathbf{I} \oplus \mathbb{R}\mathbf{S} \oplus \mathcal{A}_{\mathbf{2}}.$$

This means that every 2×2 matrix $\mathbf{K}$ can be expressed uniquely as $\mathbf{K} = \mathbf{A} + \alpha\mathbf{S} + \beta\mathbf{I}$ where $\alpha, \beta \in \mathbb{R}$. Because w is $\mathbb{R}$-linear,

$$\begin{aligned} w(\mathbf{K}) &= w(\mathbf{A}) + \alpha w(\mathbf{S}) + \beta w(\mathbf{I}) \\ &= (\mathbf{A} + \alpha\mathbf{S} + \beta\mathbf{I})\mathbf{r} \\ &= \mathbf{K}\mathbf{r} \end{aligned}$$

This proves the theorem for the case where $\Pi = \mathrm{End}(\mathbb{R}^2)$. The other cases will not be shown here, hopefully the reader has a feel for what they are like.

4 Finding the kernel

Since $\ker P_{11}$ is at most a five-dimensional subspace, there are only a handfull of families of subspaces that it can be. For convenience, let

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

We'll examine separately the cases where $\mathbf{M}$ lies in $\ker P_{11}$, and when it does not. First, suppose that

$$\begin{bmatrix} \mathbf{0} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} \in \ker P_{11}.$$

A simple calculation tells us that

$$\begin{bmatrix} \mathbf{0} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix}^{\#} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}\mathbf{u}\cdot\mathbf{v} \end{bmatrix} \in \ker P_{11} \tag{8}$$

This holds for all trace-free symmetric matrices $\mathbf{A}$. Therefore, $\mathbf{A}\mathbf{u}\cdot\mathbf{v} = 0$ for all $\mathbf{A}$ if and only if either $\mathbf{u}$ or $\mathbf{v}$ is zero. This means that that if $\mathbf{M} \notin \ker P_{11}$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. In other words, either

$$\begin{aligned} &\ker P_{11} \subset \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{u} \\ \mathbf{0} & \mathbf{u}\cdot\mathbf{r} \end{bmatrix} : \mathbf{u} \in \mathbb{R}^2 \right\} \quad \text{or} \\ &\ker P_{11} \subset \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} : \mathbf{v} \in \mathbb{R}^2 \right\}. \end{aligned} \tag{9}$$

Here, $\mathbf{r} \in \mathbb{R}^2$ is fixed, and $\rho = \mathbf{u}\cdot\mathbf{r}$ (resp. $\mathbf{v}\cdot\mathbf{r}$) because the only linear functions from $\mathbb{R}^2$ to $\mathbb{R}$ are inner products.

4.1 Case 1: $\mathbf{M} \notin \ker P_{11}$

If $\mathbf{M} \notin \ker P_{11}$, then by (9), we may assume without loss of generality that

$$\ker P_{11} = \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} : \mathbf{v} \in L \subset \mathbb{R}^2 \right\},$$

where L is the subspace containing all vectors $\mathbf{v}$. If $\dim(L) = 2$, then $L = \mathbb{R}^2$, and if $\dim(L) = 1$, then $L = \mathbb{R}\mathbf{v_0}$ for some constant vector $\mathbf{v_0}$. The approach here will first be to find all 2×2 special Jordan pencils Π_2 that $\mathbf{K}$ can belong to with each choice of L. Next, we will determine the maps w, q, and λ. Though we have previously shown that $w(\mathbf{K}) = \mathbf{Kr}$ for some constant vector $\mathbf{r}$, we still need to solve for $\mathbf{r}$ to determine Π. To start off, it is easy to show that

$$\begin{bmatrix} \mathbf{K} & w(\mathbf{K}) \\ q(\mathbf{K}) & \mathbf{0} \end{bmatrix} \# \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{K}^T\mathbf{A}\mathbf{v} & \mathbf{A}\mathbf{v}\cdot w(\mathbf{K}) \end{bmatrix} \tag{10}$$

Thus $\mathbf{K}^T\mathbf{A}\mathbf{v} \in L$ and for some fixed vector $\mathbf{r}$,

$$\mathbf{A}\mathbf{v}\cdot w(\mathbf{K}) = \mathbf{K}^T\mathbf{A}\mathbf{v}\cdot\mathbf{r}.$$

Now, we examine the cases when L has dimension two, one, and zero.

4.1.1 $\dim(L) = 2$

If L is two-dimensional, then $\mathbf{K}$ can belong to any special Jordan pencil of $(\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$. To explicitly determine the maps w, q, and λ, first we look at a matrix in the orthogonal complement of $\ker P_{11}$:

$$\begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ q(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix} \in \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} : \mathbf{v}\in\mathbb{R}^2 \right\}^{\perp}, \tag{11}$$

which implies $q(\mathbf{K})\cdot\mathbf{v} + \lambda(\mathbf{K})(\mathbf{v}\cdot\mathbf{r}) = 0$. Hence $q(\mathbf{K}) = -\lambda(\mathbf{K})\mathbf{r}$, giving us $q(\mathbf{K})$ if we can find $\lambda(\mathbf{K})$. To find $\lambda(\mathbf{K})$, we use the two-chain condition:

$$\begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ -\lambda(\mathbf{K})\mathbf{r} & \lambda(\mathbf{K}) \end{bmatrix}^{\#} = \begin{bmatrix} \mathbf{KAK} & \mathbf{KAKr} \\ -(\mathbf{K}^T\mathbf{Ar})\lambda(\mathbf{K}) & -\lambda(\mathbf{K})\mathbf{Ar}\cdot\mathbf{Kr} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \mathbf{KAK} & \mathbf{KAKr} \\ -\lambda(\mathbf{KAK})\mathbf{r} + \mathbf{v} & \lambda(\mathbf{KAK}) + \mathbf{v}\cdot\mathbf{r} \end{bmatrix}. \tag{13}$$

By equating the lower-left entries of the matrices in (12), we see that

$$\mathbf{v} = \lambda(\mathbf{KAK})\mathbf{r} - (\mathbf{K}^T\mathbf{Ar})\lambda(\mathbf{K})$$

Next, equating the lower-right entries, substituting in for v, and canceling like terms yields

$$\lambda(\mathbf{KAK})(1 + |\mathbf{r}|^2) = 0,$$

which implies that $\lambda(\mathbf{KAK}) = 0$, and hence $\lambda(\mathbf{K}) = 0$. Plugging this into (12), we get that the associated special Jordan pencil Π is

$$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} : \mathbf{K}\in\Pi_2, \mathbf{v}\in\mathbb{R}^2 \right\}. \tag{14}$$

4.1.2 $\dim(L) = 1$

If L is one-dimensional, then $L = \mathbb{R}\mathbf{v_0}$ for some constant vector $\mathbf{v_0}$, and so

$$\mathbf{K}^T\mathbf{A}\mathbf{v_0} \cdot \mathbf{r} = \mathbf{A}\mathbf{v_0} \cdot \mathbf{K}\mathbf{r}. \tag{15}$$

The only possibilities for Π_2 so that (15) is satisfied are $\mathbf{v} \otimes \mathbf{v_0}$ and $\mathbb{R}(\mathbf{a} \otimes \mathbf{v_0})$. We have

$$\begin{bmatrix} \mathbf{K} & \mathbf{K}\mathbf{r} \\ \mathbf{q}(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix} \in \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{v_0} & \mathbf{v_0} \cdot \mathbf{r} \end{bmatrix}^\perp,$$

which means that for all $\mathbf{K} \in \Pi_2$,

$$\mathbf{q}(\mathbf{K}) \cdot \mathbf{v_0} + \lambda(\mathbf{K})\mathbf{v_0} \cdot \mathbf{r} = 0.$$

There are three distinct cases, and we must consider each separately to see which 3×3 special Jordan pencils arise from each.

1. if $\mathbf{r} = \mathbf{0}$, then $\mathbf{q}(\mathbf{K}) = \mu(\mathbf{K})\mathbf{v_0}^\perp$;
2. if $\mathbf{r} \neq \mathbf{0}$, but $\mathbf{v_0} \cdot \mathbf{r} = 0$;
3. if $\mathbf{r} \neq \mathbf{0}$ and $\mathbf{v_0} \cdot \mathbf{r} \neq 0$.

We consider the three possibilities above separately.

1. If $\mathbf{r} = \mathbf{0}$, then $\mathbf{q}(\mathbf{K}) = \mu(\mathbf{K})\mathbf{v_0}^\perp$ for some function μ. Here, $\mathbf{v_0}^\perp$ denotes the unit vector orthogonal to $\mathbf{v_0}$, and $\{\mathbf{v_0}^\perp\}$ is the one-dimensional subspace containing $\mathbf{v_0}$. Then

   $$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mu(\mathbf{K})\mathbf{v_0}^\perp & \lambda(\mathbf{K}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ t\mathbf{v_0} & 0 \end{bmatrix} : t \in \mathbb{R}, \mathbf{K} \in \hat{\Pi}_2 \right\}, \tag{16}$$

   where $\hat{\Pi}_2$ denotes an allowable choice of Π_2. Now we must solve for the functions μ and λ. A simple calculations yields

   $$\begin{aligned} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mu(\mathbf{K})\mathbf{v_0}^\perp & \lambda(\mathbf{K}) \end{bmatrix}^\# &= \begin{bmatrix} \mathbf{KAK} & \mathbf{0} \\ \mu(\mathbf{K})\mathbf{K}^T\mathbf{A}\mathbf{v_0}^\perp & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{KAK} & \mathbf{0} \\ \mu(\mathbf{KAK})\mathbf{v_0}^\perp + t\mathbf{v_0} & \lambda(\mathbf{KAK}) \end{bmatrix} \end{aligned}$$

   Immediately we see that $\lambda(\mathbf{KAK}) = \mathbf{0}$, and so $\lambda(\mathbf{K}) = \mathbf{0}$. Upon comparing the lower left entries, we deduce that

   $$\mu(\mathbf{KAK}) \left|\mathbf{v_0}^\perp\right|^2 = \mu(\mathbf{K})\mathbf{A}\mathbf{v_0}^\perp \cdot \mathbf{K}\mathbf{v_0}^\perp.$$

   For both families of $\hat{\Pi}_2$, $\mathbf{K}\mathbf{v_0}^\perp = \mathbf{0}$. Therefore, $\mu(\mathbf{KAK}) = 0$, and so $\mu(\mathbf{K}) = \mathbf{0}$. This means that our solution Π is

   $$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ t\mathbf{v_0} & 0 \end{bmatrix} : \mathbf{K} \in \hat{\Pi}_2, t \in \mathbb{R} \right\}.$$

2. If $\mathbf{r} \neq \mathbf{0}$ but $\mathbf{v_0} \cdot \mathbf{r} = 0$, then for some function μ, $\mathbf{q}(\mathbf{K}) = \mu(\mathbf{K})\mathbf{r}$, and thus $\mathbf{v_0} \in \mathbf{r}^{\perp}$. As in the previous case, we know that

$$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ \mu(\mathbf{K})\mathbf{r} & \lambda(\mathbf{K}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ t\mathbf{r}^{\perp} & 0 \end{bmatrix} : \mathbf{K} \in \hat{\Pi}_2, t \in \mathbb{R} \right\},$$

and need to solve for the functions μ and λ. We shall omit the details because the process is the almost identical and it yields the same result.

3. If $\mathbf{v_0} \cdot \mathbf{r}$, then without loss of generality we may assume that $\mathbf{v_0} \cdot \mathbf{r} = -1$. In this case, $\lambda(\mathbf{K}) = q(\mathbf{K}) \cdot \mathbf{v_0}$, and so

$$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ q(\mathbf{K}) & q(\mathbf{K}) \cdot \mathbf{v_0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ t\mathbf{v_0} & -t \end{bmatrix} : \mathbf{K} \in \hat{\Pi}_2, t \in \mathbb{R} \right\} \tag{17}$$

As above, we use the 2-chain condition to find the equations that the unknown functions must satisfy:

$$\begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ \mathbf{q}(\mathbf{K}) & \mathbf{q}(\mathbf{K}) \cdot \mathbf{v_0} \end{bmatrix}^{\#} = \begin{bmatrix} \mathbf{KAK} & \mathbf{KAKr} \\ \mathbf{K}^T\mathbf{Aq}(\mathbf{K}) & \mathbf{K}^T\mathbf{Aq}(\mathbf{K}) \cdot \mathbf{r} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{KAK} & \mathbf{KAKr} \\ \mathbf{q}(\mathbf{KAK}) + t\mathbf{v_0} & \mathbf{q}(\mathbf{KAK}) \cdot \mathbf{v_0} - t \end{bmatrix}$$

Equating the lower-right entries in these two matrices allows us to solve for t. Next, we equate the lower-left entries and substitute for t, which yields

$$\mathbf{K}^T\mathbf{Aq}(\mathbf{K}) = \mathbf{q}(\mathbf{KAK}) + \mathbf{v_0}\left[\mathbf{q}(\mathbf{KAK}) \cdot \mathbf{v_0} - \mathbf{K}^T\mathbf{Aq}(\mathbf{K}) \cdot \mathbf{r}\right]$$

Taking the inner product of both sides with $\mathbf{r}$, and this simplifies to

$$\mathbf{q}(\mathbf{KAK}) \cdot (\mathbf{r} - \mathbf{v_0}) = 0.$$

Therefore, $\mathbf{q}(\mathbf{K}) = \mu(\mathbf{K})\mathbf{h}^{\perp}$. $\mathbf{h}^{\perp} = \mathbf{r}^{\perp} - \mathbf{v_0}^{\perp} = \mathbf{a_0}$.

So $\mu(\mathbf{K})\mathbf{K}^T\mathbf{Aa_0} = \mathbf{a_0}\mu(\mathbf{KAK}) + \mathbf{v_0}(\mu(\mathbf{KAK})(\mathbf{r}^{\perp} \cdot \mathbf{v_0}) - \mu(\mathbf{K})\mathbf{Aa_0} \cdot \mathbf{Kr})$. Now dot with $\mathbf{v_0}^{\perp}$:

$$\mu(\mathbf{K})\mathbf{Aa_0} \cdot \mathbf{Kv_0}^{\perp} = -(1 + |\mathbf{v_0}^{\perp}|^2)\mu(\mathbf{KAK}) \tag{18}$$

Recalling that for all $\mathbf{K} \in \hat{\Pi}_2$, $\mathbf{Kv_0}^{\perp} = \mathbf{0}$, we see that $\mu(\mathbf{K}) = 0$. Thus our solution for Π is:

$$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ t\mathbf{v_0} & t\mathbf{v_0} \cdot \mathbf{r} \end{bmatrix} : \mathbf{K} \in \hat{\Pi}_2, t \in \mathbb{R} \right\}. \tag{19}$$

The other two solutions when L is one-dimensional are simply special cases of (19), namely when $\mathbf{r} = 1$, and when $\mathbf{v_0} \cdot \mathbf{r} = 0$.

4.1.3 $L = \{\mathbf{0}\}$

If $L = \{\mathbf{0}\}$, then $\ker P_{11} = \{\mathbf{0}\}$, and so

$$\Pi = (\ker P_{11})^{\perp} = \left\{ \begin{bmatrix} \mathbf{K} & w(\mathbf{K}) \\ q(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix} : \mathbf{K} \in \Pi_2 \right\}. \tag{20}$$

a simple calculation shows that

$$\begin{bmatrix} \mathbf{K} & \mathbf{w}(\mathbf{K}) \\ q(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix}^{\#} = \begin{bmatrix} \mathbf{KAK} & \mathbf{KA}w(\mathbf{K}) \\ \mathbf{K}^T\mathbf{A}q(\mathbf{K}) & \mathbf{K}^T\mathbf{A}q(\mathbf{K}) \cdot w(\mathbf{K}) \end{bmatrix} \in \Pi$$

and thus

$$\begin{bmatrix} \mathbf{KAK} & \mathbf{KA}w(\mathbf{K}) \\ \mathbf{K}^T\mathbf{A}q(\mathbf{K}) & \mathbf{K}^T\mathbf{A}q(\mathbf{K}) \cdot w(\mathbf{K}) \end{bmatrix} = \begin{bmatrix} \mathbf{KAK} & w(\mathbf{KAK}) \\ q(\mathbf{KAK}) & \lambda(\mathbf{KAK}) \end{bmatrix}. \tag{21}$$

We have already determined the maps w, q, and λ, so the special Jordan pencil associated with this choice of L is

$$\Pi = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr_1} \\ \mathbf{K}^T\mathbf{r_2} & \mathbf{Kr_1} \cdot \mathbf{r_2} \end{bmatrix} : \mathbf{K} \in \mathbf{\Pi_2} \right\} (\mathbf{r_1}, \mathbf{r_2} \in \mathbb{R}^2). \tag{22}$$

To summarize, when $\mathbf{M} \notin \ker P_{11}$, there are three families of special Jordan pencils: the subspace (14) when $\dim(L) = 2$, (19) when $\dim(L) = 1$, and (22) when $\dim(L) = 2$.

4.2 Case 2: $\mathbf{M} \in \ker P_{11}$

The goal here is to find all possible subspaces $\ker P_{11}$ can be if it contains $\mathbf{M}$. A simple calculations yields

$$\begin{bmatrix} \mathbf{0} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} \# \begin{bmatrix} \mathbf{K} & w(\mathbf{K}) \\ q(\mathbf{K}) & \lambda(\mathbf{K}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{KAu} \\ \mathbf{K}^T\mathbf{Av} & \mathbf{Au} \cdot q(\mathbf{K}) + \mathbf{Av} \cdot w(\mathbf{K}) \end{bmatrix}, \tag{23}$$

which also belongs to $\ker P_{11}$. Using the $\mathbf{u}$ and $\mathbf{v}$ as above, let L be the suspace $(\mathbf{u}, \mathbf{v})$. Because $\mathbf{M} \in \ker P_{11}$,

$$\ker P_{11} \cong L \oplus \mathbb{R},$$

and now our problem is to find all possibile subspaces L. Equation (23) adds the restriction that

$$(\mathbf{KAu}, \mathbf{K}^T\mathbf{Av}) \subset L. \tag{24}$$

The containment in (24) holds for all $\mathbf{K} \in \Pi_2$. We shall approach this problem by considering all possible subspaces L, and then determine all possibilities of Π_2 that work with that particular L. The cases when L is 0-dimensional or 4-dimensional trivially works with all special Jordan pencils $\Pi_2 \in (\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$.

If L is one-dimensional, then

$$L = \mathbb{R}(\mathbf{u_0}, \mathbf{v_0})$$

for constant vectors $\mathbf{u_0}$ and $\mathbf{v_0}$. If $\mathbf{u_0} \neq \mathbf{0}$, then $\{\mathbf{Au_0}\}$ spans $\mathbb{R}^2$. Therefore, every $\mathbf{K} \in \Pi_2$ must satisfy $\mathbf{Kx} = \lambda \mathbf{u_0}$ for every $\mathbf{x} \in \mathbb{R}^2$. This forces

$$\Pi_2 = \{\mathbf{u_0} \otimes \mathbf{v} \ : \ \mathbf{v} \in \mathbb{R}^2\}.$$

By symmetry, the subspace $\Pi_2 = \{\mathbf{v} \otimes \mathbf{u_0} \ : \ \mathbf{v} \in \mathbb{R}^2\}$ also works.

The case where L is three-dimensional is also straight-forward. A three-dimensional subspace of $\mathbb{R}^4$ can be described by the equation

$$L = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 : \mathbf{u} \cdot \mathbf{u_0} + \mathbf{v} \cdot \mathbf{v_0} = 0\}, \tag{25}$$

where $\mathbf{u_0}$ and $\mathbf{v_0}$ are constant vectors in $\mathbb{R}^2$. For any $(\mathbf{u}, \mathbf{v}) \subset L$, we have $\mathbf{KAu} \cdot \mathbf{u_0} + \mathbf{K}^T\mathbf{Av} \cdot \mathbf{v_0} = 0$, or equivalently,

$$\mathbf{u} \cdot \mathbf{AK}^T\mathbf{u_0} + \mathbf{v} \cdot \mathbf{AKv_0} = 0.$$

Therefore, $(\mathbf{u}, \mathbf{v})$ is orthogonal to the subspace generated by $(\mathbf{AK}^T\mathbf{u_0}, \mathbf{AKv_0})$. If $\mathbf{K}^T\mathbf{u_0} \neq \mathbf{0}$ then $\{\mathbf{AK}^T\mathbf{u_0}\}$ spans $\mathbb{R}^2$. It follows that $\mathbf{K}^T\mathbf{u_0} = \mathbf{0}$, and by a similar argument, $\mathbf{Kv_0} = \mathbf{0}$. To summarize, the subspaces Π_2 that work with three-dimensional L as defined in (25) are as follows.

1. $\Pi_2 = \mathbb{R}(\mathbf{u_0}^\perp \otimes \mathbf{v_0}^\perp)$ when $\mathbf{u_0}$, $\mathbf{v_0} \neq \mathbf{0}$;
2. $\Pi_2 = \{\mathbf{u_0}^\perp \otimes \mathbf{v} : \mathbf{v} \in \mathbb{R}^2\}$ or $\Pi_2 = \mathbb{R}(\mathbf{u_0}^\perp \otimes \mathbf{a})$ when $\mathbf{u_0} \neq \mathbf{0}$, $\mathbf{v_0} = \mathbf{0}$;
3. $\Pi_2 = \{\mathbf{v} \otimes \mathbf{v_0} : \mathbf{v} \in \mathbb{R}^2\}$ or $\Pi_2 = \mathbb{R}(\mathbf{a} \otimes \mathbf{v_0}^\perp)$ when $\mathbf{u_0} = \mathbf{0}$, $\mathbf{v_0} \neq \mathbf{0}$.

This characterizes all possibilities for Π_2 when L is three-dimentional. We now turn our attention to the most complicated case; when L is be two-dimensional. In this case, there are three possibilities for L:

(*i*) $L = Span\{(\mathbf{u_0}, \mathbf{0}), (\mathbf{0}, \mathbf{v_0})\}$, $\mathbf{u_0} \neq \mathbf{0}$, $\mathbf{v_0} \neq \mathbf{0}$.

(*ii*) $L = \{(\mathbf{u}, \mathbf{M_0 u}) : \mathbf{u} \in \mathbb{R}^2\}$, $\mathbf{M_0} \in \mathrm{End}(\mathbb{R}^2)$ is fixed.

(*iii*) $L = \{(\mathbf{M_0 v}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^2\}$, $\mathbf{M_0} \in \mathrm{End}(\mathbb{R}^2)$ is fixed.

A trivial case is when $\Pi_2 = \{\mathbf{0}\}$, which satisfies all three possibilities for L. If L is as given by (*i*), then the only non-empty Π_2 that works is $\Pi_2 = \mathbb{R}(\mathbf{u_0} \otimes \mathbf{v_0})$. All that remains it to determine L for case (*ii*), because then it easy to solve (*iii*). This is because if Π_2 works with $L = \{(\mathbf{u}, \mathbf{v})\} \subset \mathbb{R}^4$, then Π_2^T works with $L^T = \{(\mathbf{v}, \mathbf{u})\} \subset \mathbb{R}^4$.

By (24), any L of the form in (ii) has the condition that $(\mathbf{KAu}, \mathbf{K}^T\mathbf{AM_0u}) \in L$, which means that

$$\mathbf{K}^T\mathbf{AM_0u} = \mathbf{M_0KAu}$$

for all $\mathbf{u} \in \mathbb{R}^2$, from which we conclude that $\mathbf{M_0}$ satisfies

$$\mathbf{K}^T\mathbf{AM_0} = \mathbf{M_0KA} \tag{26}$$

All that remains is to solve for $\mathbf{M_0}$ for each family of 2×2 special Jordan pencils. Our approach makes use of the complex variable representations of 2×2 matrices, specifically the maps $\phi, \psi : \mathbb{C} \to \mathrm{End}(\mathbb{R}^2)$ defined by:

$$\phi : a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \psi : a + bi \mapsto \begin{bmatrix} a & b \\ b & -a \end{bmatrix}. \tag{27}$$

We shall also use the canonical map $\pi : \mathbb{C} \to \mathbb{R}^2$ that sends a complex number $a + bi$ to the vector (a, b). These three maps have the following easily verified properties. For any u , $v \in \mathbb{C}$,

1. $\phi(u)\phi(v) = \phi(uv)$ 2. $\phi(u)\psi(v) = \psi(uv)$ 3. $\psi(u)\psi(v) = \phi(u\overline{v})$
4. $\psi(u)\phi(v) = \psi(u\overline{v})$ 5. $\phi(u)\pi(v) = \pi(uv)$ 6. $\psi(u)\pi(v) = \pi(u\overline{v})$

The maps ϕ and ψ will retain their definitions as above for the remainder of this paper, but we will often omit writing the map π for clarity of notation.

We now resume our discussion of solving (26) for each choice of parent solution Π_2.

- If $\Pi_2 = \{\mathbf{0}\}$ then any subspace L works.
- If $\Pi_2 = \mathbb{R}(\mathbf{a} \otimes \mathbf{b})$, then for all $\mathbf{A} \in \mathcal{A}_2$, $(\mathbf{b} \otimes \mathbf{a})\mathbf{AM_0} = \mathbf{M_0}(\mathbf{a} \otimes \mathbf{b})\mathbf{A}$, which is the same as

  $$\mathbf{b} \otimes \mathbf{M_0}^T\mathbf{Aa} = \mathbf{M_0a} \otimes \mathbf{Ab}.$$

  Since $\mathbf{A} = \psi(z)$ for some $\mathbf{z} \in \mathbb{C}$,

  $$\mathbf{Aa} = \pi(\mathbf{z}\overline{\mathbf{a}})\phi(\overline{\mathbf{a}})\mathbf{z}.$$

  Thus for all $\mathbf{z} \in \mathbb{C}$, $\mathbf{b} \otimes \mathbf{M_0}^T\phi(\overline{\mathbf{a}})\mathbf{z} = \mathbf{M_0a} \otimes \phi(\overline{\mathbf{b}})\mathbf{z}$, which is equivalent to

  $$\mathbf{b} \otimes \phi(1/\overline{b})\mathbf{M_0}^T\phi(\overline{a})\mathbf{z} = \mathbf{M_0a} \otimes \mathbf{z}.$$

  If $\mathbf{M_0a} = \mathbf{0}$ then

  $$\mathbf{b} \otimes \phi(\mathbf{1}/\overline{\mathbf{b}})\mathbf{M_0}^T\phi(\overline{\mathbf{a}})\mathbf{z} = \mathbf{0}$$

  for all $\mathbf{z} \in \mathbb{R}^2$, and therefore

  $$\phi(1/\overline{\mathbf{b}})\mathbf{M_0}^T\phi(\overline{\mathbf{a}}) = \mathbf{0}$$

and hence $\mathbf{M_0} = \mathbf{0}$. If $\mathbf{M_0 a} \neq \mathbf{0}$ then for all $\mathbf{z} \in \mathbb{R}^2$, $\mathbf{z}$ is an eigenvector of $\phi(1/\overline{\mathbf{b}})\mathbf{M_0}^T\phi(\overline{\mathbf{a}})$. Thus

$$\phi(1/\overline{\mathbf{b}})\mathbf{M_0}^T\phi(\overline{\mathbf{a}}) = \lambda\mathbf{I}$$

which implies that $\mathbf{M_0}^T = \lambda\phi(\overline{\mathbf{b}}/\overline{\mathbf{a}})$, and so $\mathbf{M_0} = \lambda'\phi(\overline{\mathbf{a}}\mathbf{b})$.

- If $\Pi_2 = \{\mathbf{v} \otimes \mathbf{b} : \mathbf{v} \in \mathbb{R}^2\}$, then $\mathbf{M_0} = \lambda(v)\phi(\overline{\mathbf{v}}\mathbf{b})$ for all $\mathbf{v} \in \mathbb{R}^2$, which implies that $\mathbf{M_0} = \mathbf{0}$.
- If $\Pi_2 = \mathcal{A}_2$, then $\mathbf{M_0} = \phi(\mathbf{c})$ for some $\mathbf{c} \in \mathbb{C}$.
- If $\Pi_2 = \Pi_\beta$, a straightforward but tedious computation using (26) gives

$$\mathbf{M_0} = \lambda \begin{bmatrix} 1 & -1/\beta \\ 1/\beta & 1 \end{bmatrix}.$$

- If $\Pi_2 = \mathrm{End}(\mathbb{R}^2)$ then

$$\mathbf{M_0} = \lambda(\beta) \begin{bmatrix} 1 & -1/\beta \\ 1/\beta & 1 \end{bmatrix}$$

for all $\beta \in \mathbb{R}$, where λ is a constant that depends on β. therefore, $\mathbf{M_0} = \mathbf{0}$.

These are all the posibilities for L when L is two-dimensional. We now have a complete list of the possible subspaces L, which are given in Table 4.2.

Recall that our goal in Case 2 thus far has been to determine all subspaces $\ker P_{11}$ that contain $\mathbf{M}$. We know that $\ker P_{11} \cong L \oplus \mathbb{R}$, and we just found all eighteen possiblities for L, hence determining $\ker P_{11}$. To compute the special Jordan pencil, associated with each possible $\ker P_{11}$, we must examine it individually and determine what the maps w and q, that show up in the orthogonal complement of $\ker P_{11}$. We have shown that $w(\mathbf{K}) = \mathbf{Kr}$ for a constant vector $\mathbf{r}$, but we must determine $\mathbf{r}$ explicitly for each of the eighteen cases. However, this requires many tedious calculations which are omitted from this paper. The results are summarized in the next section.

5 The eleven parent solutions

The three subspaces found in Case 1, along with the eighteen found in Case 2 make up all special Jordan pencils of $\mathrm{End}(\mathbb{R}^2)$ with $\mathcal{A}$ defined as in (3). However, many of these subspaces are special cases or intersections of other solutions. In fact, the list of twenty-one subspaces has been simplified to a list of eleven families of parent solutions because many were just special cases, intersections and transposes of other special Jordan pencils. Every 3×3 special Jordan pencil

Π_2	possible subspaces $L \subset \mathbb{R}^4$
$\mathrm{End}(\mathbb{R}^2)$	$\{\mathbf{0}\}$, $\mathbb{R}^4$, $(\mathbb{R}^2, \mathbf{0})$, $(\mathbf{0}, \mathbb{R}^2)$
Π_β	$\{\mathbf{0}\}$, $\mathbb{R}^4$, $\{(\mathbf{u}, \mathbf{M_0 u}) : \mathbf{u} \in \mathbb{R}^2\}$, where $\mathbf{M_0} = \lambda \begin{bmatrix} 1 & -1/\beta \\ 1/\beta & 1 \end{bmatrix}$
$\mathcal{A}_2$	$\{\mathbf{0}\}$, $\mathbb{R}^4$, $\{(\mathbf{u}, \mathbf{M_0 u}) : \mathbf{u} \in \mathbb{R}^2\}$, where $\mathbf{M_0} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$
$\mathbf{v} \otimes \mathbf{a}$	$\{\mathbf{0}\}$, $\mathbb{R}^4$, $\mathbb{R}(\mathbf{0}, \mathbf{a})$, $\{(\mathbf{u}, \mathbf{0}) : \mathbf{u} \in \mathbb{R}^2\}$, $\{(\mathbf{0}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^2\}$, $\{(\mathbf{u}, \mathbf{v}) : \mathbf{v} \cdot \mathbf{a}^\perp = 0\}$, $\{(\mathbf{u}, t\mathbf{a}) : \mathbf{u} \in \mathbb{R}^2, t \in \mathbb{R}\}$
$\mathbb{R}(\mathbf{a} \otimes \mathbf{b})$	$\{\mathbf{0}\}$, $\mathbb{R}^4$, $\mathbb{R}(\mathbf{a}, \mathbf{b})$, $\mathbb{R}(\mathbf{0}, \mathbf{b})$, $\mathbb{R}(\mathbf{a}, \mathbf{0})$, $\{(\mathbf{u}, \lambda\phi(\mathbf{ab})^T\mathbf{u}) : \mathbf{u} \in \mathbb{R}^2\}$ with $\lambda \in \mathbb{R}$ fixed , $\{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \cdot \mathbf{a}^\perp + \mathbf{v} \cdot \mathbf{b}^\perp = 0\}$, $\{(\mathbf{u}, \lambda\mathbf{b}) : \mathbf{u} \in \mathbb{R}^2, \lambda \in \mathbb{R}\}$ $\{(t\mathbf{a}, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^2, t \in \mathbb{R}\}$, $Span\{(\mathbf{a}, \mathbf{0}), (\mathbf{0}, \mathbf{b})\}$

Table 1: Compatible Pairs Π_2 , L

is either in one of the families of parent solutions listed below, or can be obtained by intersections of the parent solutions, and/or by taking transposes.

$$\Pi^{(1)}_{\Pi_2,\mathbf{r}} = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{Kr} \\ \mathbf{v} & \mathbf{v}\cdot\mathbf{r} \end{bmatrix} : \mathbf{K} \in \Pi_2 \right\} (\mathbf{r} \in \mathbb{R}^2)$$

$$\Pi^{(2)}_{\mathbf{a}} = \left\{ \begin{bmatrix} \mathbf{u}\otimes\mathbf{a} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} : \mathbf{u},\mathbf{v} \in \mathbb{R}^2, \rho \in \mathbb{R} \right\} (\mathbf{a} \neq \mathbf{0})$$

$$\Pi^{(3)}_{\mathbf{a},\mathbf{b}} = \left\{ \begin{bmatrix} s\mathbf{a}\otimes\mathbf{b} & t\mathbf{a} \\ \mathbf{v} & \rho \end{bmatrix} : s,t,\rho \in \mathbb{R}^2, \mathbf{v} \in \mathbb{R}^2 \right\} (\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0})$$

$$\Pi^{(4)}_{\Pi_2} = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{u} \\ \mathbf{v} & \rho \end{bmatrix} : \mathbf{K} \in \Pi_2, \mathbf{u},\mathbf{v} \in \mathbb{R}^2, \rho \in \mathbb{R} \right\}$$

$$\Pi^{(5)}_{\Pi_2,\mathbf{r}} = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{u} \\ \mathbf{K}^T\mathbf{r} & \rho \end{bmatrix} : \mathbf{K} \in \Pi_2, \mathbf{u} \in \mathbb{R}^2, \rho \in \mathbb{R} \right\} (\mathbf{r} \in \mathbb{R}^2)$$

$$\Pi^{(6)}_{\beta,\mathbf{r}} = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{v} \\ \mathbf{K}^T\mathbf{r} + \mathbf{M_0}\mathbf{v} & \rho \end{bmatrix} : \mathbf{K} \in \Pi_\beta, \mathbf{v} \in \mathbb{R}^2, \rho \in \mathbb{R} \right\}$$
$$(\mathbf{r} \in \mathbb{R}^2, \mathbf{M_0} = \lambda \begin{bmatrix} 1 & -1/\beta \\ 1/\beta & 1 \end{bmatrix})$$

$$\Pi^{(7)}_{\mathbf{a},\mathbf{b}} = \left\{ \begin{bmatrix} s(\mathbf{a}\otimes\mathbf{b} & \mathbf{u}) \\ \mathbf{v} & \rho \end{bmatrix} : s,\rho \in \mathbb{R}, \mathbf{u}\cdot\mathbf{a}^\perp + \mathbf{v}\cdot\mathbf{b}^\perp = 0 \right\} (\mathbf{a},\mathbf{b} \in \mathbb{R}^2)$$

$$\Pi^{(8)}_{\mathbf{a},\mathbf{b},m} = \left\{ \begin{bmatrix} s(\mathbf{a}\otimes\mathbf{b}) & t\mathbf{a} \\ (t-ms)\mathbf{b} & \rho \end{bmatrix} : s,t,\rho \in \mathbb{R} \right\} (m \in \{-1,0,1\}, \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0})$$

$$\Pi^{(9)}_{\mathbf{a}} = \left\{ \begin{bmatrix} \mathbf{v}\otimes\mathbf{a} & \mathbf{u} \\ t\mathbf{a} & \rho \end{bmatrix} : \mathbf{u},\mathbf{v} \in \mathbb{R}^2, t,\rho \in \mathbb{R} \right\} (\mathbf{a} \neq \mathbf{0})$$

$$\Pi^{(10)}_{c,\mathbf{r}} = \left\{ \begin{bmatrix} \mathbf{K} & \mathbf{u} - \phi(\mathbf{c})\mathbf{Kr} \\ \phi(\mathbf{c})\mathbf{u} + \mathbf{Kr} & \rho \end{bmatrix} : \mathbf{K} \in \mathcal{A}_2, \mathbf{u} \in \mathbb{R}^2, \rho \in \mathbb{R} \right\}$$
$$(\mathbf{r} \in \mathbb{R}^2, \mathbf{c} \in \mathbb{C})$$

$$\Pi^{(11)}_{\mathbf{a},\mathbf{b}} = \left\{ \begin{bmatrix} t(\mathbf{a}\otimes\mathbf{b}) & \mathbf{u} \\ t\mathbf{b} + \phi(\bar{\mathbf{a}}\mathbf{b})\mathbf{u} & \rho \end{bmatrix} : \mathbf{u} \in \mathbb{R}^2, t,\rho \in \mathbb{R} \right\} (\mathbf{a},\mathbf{b} \in \mathbb{R}^2 \setminus \{\mathbf{0}\})$$

6 Further research

The theory of special Jordan pencils of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$ arose from research about composite materials. One natural question is which subspaces are special Jordan pencils of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A}_3)$, where $\mathcal{A}_3$ is the space of all 3×3 symmetric trace-free matrices. Because $\mathcal{A} \subset \mathcal{A}_3$, any special Jordan pencil of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$ must also be a special Jordan pencil of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$.

Another interesting questions that arises is how the theory of special Jordan pencils generalize to higher dimensions, both for $\mathcal{A}_n$, and for the subspace of matrices presented in this paper; symmetric, trace-free and zero everywhere except the upper-left 2×2 block. Unfortunately, the approach layed out in this paper most likely will not work to determine all special Jordan pencils of $\mathrm{End}(\mathbb{R}^4)$. Finding all special Jordan pencils of $\mathrm{End}(\mathbb{R}^4)$ appears to be a huge task, even with the aid of a computer. There might be another way to find all subspaces, but it is not known. It would be desirable to find a better method, because special Jordan pencils of larger spaces do arise in the theory of composite materials.

Finally, there is a natural generalization of the two-chain condition. A subspace Π satisfies the *three-chain condition* if

$$\mathbf{K_1 A_1 K_2 A_2 K_3} + \mathbf{K_3 A_2 K_2 A_1 K_1} \in \Pi$$

for all $\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3} \in \Pi$ and all $\mathbf{A_1}, \mathbf{A_2} \subset \mathcal{A}$. Of course, this notion can further be generalized into the n-chain condition. Using the aid of a computer, it has been determined that all special Jordan pencils of $(\mathrm{End}(\mathbb{R}^2), \mathcal{A}_2)$ that satisfy the two-chain condition satisfy both the three-chain and the four-chain condition. The same holds for special Jordan pencils of $(\mathrm{End}(\mathbb{R}^3), \mathcal{A})$. There is no proof of this fact, but in every case it happens to be true. This is quite a remarkable property, because there the higher n-chain properties are not equivalent to the property defined in (2), which is used extensively in this paper to determine special Jordan pencils. However, it is not known whether or not a subspace that satisfies the n-chain condition will satisfy the $(n+1)$-chain condition, or even of the two-chain condition implies the three-chain condition for special Jordan pencils of higher dimensional spaces.

Algebras Satisfied by the Parent Solutions

Subspace Π	3-chain $\mathcal{A}$	4-chain $\mathcal{A}$	2-chain $\mathcal{A}_3$	3-chain $\mathcal{A}_3$	4-chain $\mathcal{A}_3$
$\Pi^{(1)}_{r,End(\mathbb{R}^2)}$	Yes	Yes	Yes	Yes	Yes
$\Pi^{(1)}_{r,\Pi_\beta}$	Yes	Yes	No	No	No
$\Pi^{(1)}_{r,a\otimes v}$	Yes	Yes	Yes	Yes	Yes
$\Pi^{(1)}_{r,v\otimes a}$	Yes	Yes	No	No	No
$\Pi^{(2)}_{a}$	Yes	Yes	No	No	No
$\Pi^{(3)}_{a,b}$	Yes	Yes	No	No	No
$\Pi^{(4)}_{r,End(\mathbb{R}^2)}$	Yes	Yes	Yes	Yes	Yes
$\Pi^{(4)}_{r,\Pi_\beta}$	Yes	Yes	No	No	No
$\Pi^{(4)}_{r,a\otimes v}$	Yes	Yes	No	No	No
$\Pi^{(4)}_{r,v\otimes a}$	Yes	Yes	No	No	No
$\Pi^{(5)}_{r,End(\mathbb{R}^2)}$	Yes	Yes	No	No	No
$\Pi^{(5)}_{r,\Pi_\beta}$	Yes	Yes	No	No	No
$\Pi^{(5)}_{r,a\otimes v}$	Yes	Yes	No	No	No
$\Pi^{(5)}_{r,v\otimes a}$	Yes	Yes	No	No	No
$\Pi^{(6)}_{r,\lambda,\beta}$	Yes	Yes	No	No	No
$\Pi^{(7)}_{a,b}$	Yes	Yes	No	No	No
$\Pi^{(8)}_{a,b,m}$	Yes	Yes	No	No	No
$\Pi^{(9)}_{a}$	Yes	Yes	Yes	Yes	Yes
$\Pi^{(10)}_{c,r}$	Yes	Yes	No	No	No
$\Pi^{(11)}_{a,b}$	Yes	Yes	No	No	No

References

[1] Y. Grabovsky, 2×2 *Special Jordan Pencils: An Example.*